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Heterogeneous voter models

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We introduce the *heterogeneous voter model* (HVM), in which each agent has its own intrinsic rate to change state, reflective of the heterogeneity of real people, and the *partisan voter model* (PVM), in which each agent has an innate and fixed preference for one of two possible opinion states. For the HVM, the time until consensus is reached is much longer than in the classic voter model. For the PVM in the mean-field limit, a population evolves to a preference-based state, where each agent tends to be aligned with its internal preference. For finite populations, discrete fluctuations ultimately lead to consensus being reached in a time that scales exponentially with population size.

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The paradigmatic voter model [1] describes the evolution toward consensus in a population of agents that possess a discrete set of opinions. In a single update, a random voter is picked and it adopts the opinion state of a randomly selected neighbor. By repeated updates, a finite and initially diverse population necessarily reaches consensus in a time that typically scales as a power law of the population size N [1,2]. In many respects, the voter model resembles the kinetic Ising model with zero-temperature Glauber dynamics. Because of this connection to nonequilibrium spin systems [3] and the utility of the voter model for interacting particle [1] and social [4] systems, the voter model is widely studied in the physics literature (see, e.g., [5–8]). In this work, we generalize the traditional voter model in two simple, but far-reaching ways to incorporate the heterogeneity of real people [9]:

(i) *Heterogeneous voter model* (HVM): each voter has an intrinsic and distinct "flip" rate.

(ii) *Partisan voter model* (PVM): each voter has an innate and fixed preference for one opinion state.

The role of heterogeneity was emphasized in classic work by Granovetter [10], in which collective social behavior is determined by the diversity of individual thresholds to act in response to stimuli. In the context of the voter model, heterogeneity has been studied in the extreme situation where some voters are "zealots" that never change opinion [11,12]. This attribute prevents consensus from being reached when zealots with different opinions exist. In our HVM, the flip rate of each agent is taken from a continuous distribution that excludes zero. Since every voter can, in principle, change state, a finite system necessarily reaches consensus, albeit slowly. For the PVM, the innate voting preference of each agent leads to a collective state in which the opinion of each voter tends to align with its own preference. This competition between self-interest and consensus has been modeled previously [13], and was the focus of recent social experiments that attempted to elucidate the role of the preference strength on the dynamics [14]. Here we investigate basic properties of these two models from a statistical physics perspective.

Heterogeneous voter model (HVM): Each agent can be in one of two opinion states that we label as **0** and **1**. We first determine the exit probability $E(\rho)$ that a finite population with initial density ρ of **1** voters ends with **1** consensus. Because the average density ρ of **1** voters is conserved for the classic voter model on regular networks [1], the final density of **1** voters, which equals $0\times[1-E(\rho)]+1\times E(\rho)=E(\rho)$, must equal the initial density ρ.

To derive the exit probability for the HVM, we need to construct an analogous conservation law. Let $\eta(x)=0,1$ denote the state of a voter at node x in a social network, η the state of all voters in the system, and η^x the system state derived from η when only the voter at x flips. The transition probability of a voter at node x is given by

$$\mathbf{P}[\eta\to\eta^x]=\sum_y \frac{r_x}{Nk}[\Phi(x,y)+\Phi(y,x)], \tag{1}$$

where y are the neighbors of node x, r_x is the intrinsic flip rate of the voter at x, and k is the number of neighbors of each node in a regular network. The factor $\Phi(x,y)\equiv\eta(x)[1-\eta(y)]$ guarantees that voters at x and y have different opinions so that an update actually occurs. The transition probability [Eq. (1)] corresponds to the *invasion process* on a heterogeneous network [see Eqs. (4) and (5) in [8]], in which a randomly selected agent imposes its state on a neighbor; in the complementary voter model the agent imports the state of a neighbor.

The average change in $\eta(x)$ equals the difference between the probabilities that $\eta(x)$ changes from 0 to 1 and from 1 to 0. Thus $\langle\Delta\eta(x)\rangle=[1-2\eta(x)]\mathbf{P}[\eta\to\eta_x]$. Using the transition rate [Eq. (1)], it is immediate to see that the factor r_x leads to $\langle\eta(x)\rangle$ not being conserved. By construction, however, the rate-weighted density of **1** voters,

$$\omega\equiv\frac{\sum_x \eta(x)/r_x}{\sum_x 1/r_x}, \tag{2}$$

is conserved in the HVM. Thus the probability for a system with initial value ω to reach **1** consensus equals ω. As a consequence, a tiny fraction of very stubborn **1** voters (those with flip rates $r \ll 1$) leads to a probability of reaching **1** consensus that is arbitrarily close to one.

To determine the average consensus time $\langle T_N \rangle$ for a population of N heterogeneous voters, we focus on the distribution of intrinsic rates $p(r)=Ar^{-\alpha}$, with $r\in(0,r_+]$ and α in the range $[0,1)$ so the distribution is normalizable. For convenience in comparing cases with different α, we fix the average flip rate of the entire population $\langle r \rangle = 1$. These conditions give $r_+ = \frac{2-\alpha}{1-\alpha}$ and $A=(2-\alpha)r_+^{\alpha-2}$. Although the lower limit of the flip rate distribution is zero, the smallest rate r_- among a finite population of N voters is nonzero and is determined by the extremal criterion [15]

$$\int_0^{r_-} Ar^{-\alpha}dr = N^{-1}, \tag{3}$$

which gives $r_- \sim N^{-1/(1-\alpha)}$. As we shall see, these stubbornest voters control the consensus time.

We take the initial condition that each voter is independently in the **0** or the **1** state with probability $\frac{1}{2}$. For voters on a complete graph of $N \gg 1$ nodes, we now follow the analysis of the closely related invasion process on heterogeneous networks [8]. We partition voters according to their flip rates and denote by ρ_r the density of **1** voters that have flip rate in the range $[r, r+\Delta r]$. The evolution of ρ_r is governed by a Fokker-Planck equation whose drift velocity drives each of the densities ρ_r to the common value ρ in a convergence time scale that is of the order of $1/r$. Subsequently ρ evolves in the same manner as the homogeneous voter model on the complete graph with an *effective* population size $N_{\text{eff}}=N\langle 1/r\rangle$. Because the consensus time of the classic voter model on the complete graph is proportional to this effective size, we obtain, for the HVM,

$$\langle T_N \rangle \sim N\langle 1/r \rangle. \tag{4}$$

Heterogeneity hinders the approach to consensus because $\langle 1/r \rangle > 1/\langle r \rangle$. The dependence of Eq. (4) arises because a voter with flip rate r effectively corresponds to $1/r$ voters with flip rate 1. For the power-law distribution of flip rates $p(r)=Ar^{-\alpha}$, Eq. (4), in conjunction with $r_- \sim N^{-1/(1-\alpha)}$, yields $\langle 1/r \rangle \sim N^{\alpha/(1-\alpha)}$. Thus

$$\langle T_N \rangle \sim \begin{cases} N \ln N & \alpha = 0, \\ N^{1/(1-\alpha)} & 0 < \alpha < 1, \end{cases} \tag{5}$$

in agreement with simulation results (Fig. 1). Notice that the convergence time for the stubbornest voters, $1/r_- \sim N^{1/(1-\alpha)}$, is of the same order as the consensus time; evidently, this subtlety does not affect our simulation results. Finally, if the lower limit of the distribution of flip rates is strictly greater than zero, then the mean consensus time is linear in N.

In one dimension, the HVM organizes into alternating domains of like-minded voters at long times, and consensus is reached when all the intervening (and mobile) domain walls annihilate. This complete annihilation occurs when a single domain wall explores on the order of N nodes. Thus consider the motion of a single domain wall between nodes $i-1$ and i—all voters to the left of i are in state **0** and all other voters

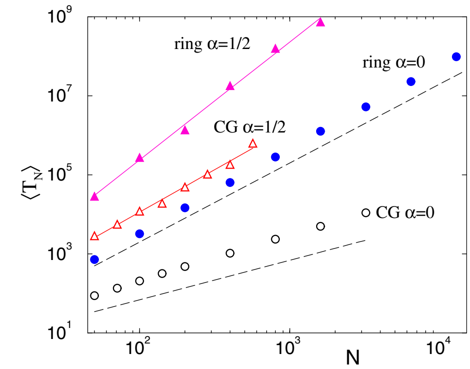

FIG. 1. (Color online) Average consensus time $\langle T_N \rangle$ for 10^4 realizations of the HVM on (a) a complete graph (CG) of N nodes (open symbols), and (b) ring of N nodes (filled symbols), with circles and triangles corresponding to $\alpha=0$ and $\frac{1}{2}$. The solid lines are power-law data fits, with slopes 2.03 for the CG and 2.98 for the ring, compared to the values 2 and 3 from theory [Eq. (5) and immediately after Eq. (8)]. The dashed lines are the exact results for the homogeneous voter model: (a) $T_N = N \ln 2$ on the CG and (b) $T_N \sim N^2$ on the ring.

are in state **1**. In a time interval dt, the probabilities that this domain wall hops one step to the right and to the left are, respectively,

$$p_i = r_i \Delta t, \quad q_i = r_{i-1}\Delta t. \tag{6}$$

The crucial point is that hopping probabilities at adjacent nodes are anti-correlated—if the bias at node i is to the right (corresponding to $r_i > r_{i-1}$), then it is more likely that $r_{i+1} < r_i$ and the bias at node $i+1$ will be to the left. More precisely, for three consecutive rates (r_{i-1}, r_i, r_{i+1}) with the constraint $r_i > r_{i-1}$, their relative sizes may equiprobably be SML, SLM, or MLS, where S, M, L denotes the smallest, middle, and largest of these three rates. The latter two cases correspond to a leftward bias between nodes i and $i+1$, which thus occurs with probability 2/3.

With the hopping probabilities q_i and p_i, the mean first-passage time τ for a particle to travel from $i=0$ to $i=N$ in the finite interval $[0,N]$ is known [16–18],

$$\tau = \sum_{k=0}^{N-1} \frac{1}{p_k} + \sum_{k=0}^{N-2} \frac{1}{p_k} \sum_{i=k+1}^{N-1} \prod_{j=k+1}^{i} \frac{q_j}{p_j}. \tag{7}$$

In the Sinai problem [19], where the p_i and q_i are independent, identically distributed random variables, τ grows as e^N [16]. For the HVM, the anticorrelated hopping probabilities [Eq. (6)] lead to substantial cancellations in the above product and yields

$$\tau = N\left\langle \frac{1}{r} \right\rangle + \frac{(N-1)N}{2}\left\langle \frac{1}{r} \right\rangle. \tag{8}$$

Using our previous result for $\langle 1/r \rangle$, we thus obtain $\langle T_N \rangle \sim N^{2+\alpha/(1-\alpha)}$, which agrees well with our numerical simulations shown in Fig. 1(b).

Partisan voter model (PVM). Without being pejorative, define state **1** as "democrat" and state **0** as "republican." In the PVM, each voter has a fixed and innate preference for democrat or republican. Equivalently, each voter experiences its own random field. A voter can therefore exist in one of four states: a "concordant democrat" is a democratic voter in its preferred **1** state, while "discordant democrat" is a democratic voter that happens to be in the **0** state. Complementary definitions apply for "concordant republican" and "discordant republican."

Denote the densities of these four types of voters as D_c, D_d, R_c, and R_d, respectively. The density of voters ρ that happen to be in the **1** state (current democrats) is $\rho = D_c + R_d$. In a single update event, a voter in a social network is randomly selected and it selects a random neighbor. If these two voters are in the same state, nothing happens. If the pair is in different opinion states, the initial voter changes its state as follows:

(i) If the voter becomes aligned with its preference, the change occurs at rate $1+\epsilon$.

(ii) If the voter becomes anti-aligned with its preference, the change occurs at rate $1-\epsilon$.

Thus ϵ quantifies the strength of the intrinsic preference, or partisanship. If $\epsilon = 1$, each voter becomes a zealot that never changes opinion after aligning with its innate preference, while $\epsilon = 0$ recovers the classic voter model. A similar dichotomous rate arises for catalysis on a disordered surface [20], where surface heterogeneity controls the adsorption rate of different reactants on the surface.

By analyzing the outcomes from all possible pairs of opposite-opinion voters, the rate equations for the densities D_c and D_d in the mean-field limit are

$$\dot{D}_c = 2\epsilon D_c D_d + (1+\epsilon) D_d R_d - (1-\epsilon) D_c R_c,$$

$$\dot{D}_d = -2\epsilon D_c D_d + (1-\epsilon) D_c R_c - (1+\epsilon) D_d R_d. \tag{9}$$

The equations for R_c and R_d are obtained from Eq. (9) by interchanging $R \leftrightarrow D$. Note that $\dot{D}_c + \dot{D}_d = \dot{R}_c + \dot{R}_d = 0$, which expresses the conservation of voters of any type.

Let D and R denote the density of intrinsic democrats and republicans, respectively. For simplicity, we specialize to the symmetric case of $D = R = \frac{1}{2}$, so that the density of democrats of any kind, concordant and discordant, is given by $D_c + D_d = D = \frac{1}{2}$; similarly, $R_c + R_d = \frac{1}{2}$. Using these relations, $\rho = D_c + \frac{1}{2} - R_c \equiv \frac{1}{2} + \Delta$. In terms of the sum $\Sigma \equiv D_c + R_c$ and the difference $\Delta \equiv D_c - R_c$ in the densities of concordant voters, Eqs. (9) simplify to

$$\dot{\Delta} = \epsilon\Delta - 2\epsilon\Sigma\Delta,$$

$$\dot{\Sigma} = \frac{1}{2}(1+\epsilon) - \Sigma - 2\epsilon\Delta^2. \tag{10}$$

For $\epsilon = 0$ (classic voter model), $\dot{\Delta} = \dot{\rho} = 0$, and the average density of voters in either opinion state is conserved. Because $\dot{\Sigma} = \frac{1}{2}(1-2\Sigma)$, the density of concordant voters is driven to $\frac{1}{2}$ in the final consensus state. For general $0 < \epsilon < 1$, there are two fixed points (Fig. 2):

(i) *Self-centered* (S): $\Delta^* = 0$ and $\Sigma^* = \frac{1}{2}(1+\epsilon)$. Each voter tends to internal concordance at the expense of consensus.

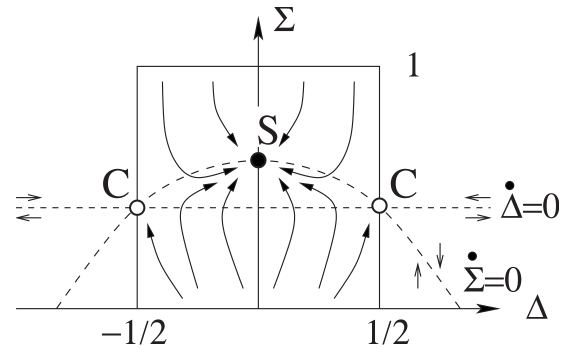

FIG. 2. Flow diagram (schematic) in the physical portion of the Δ-Σ phase plane (inside square). The open circles denote the fixed (saddle) points that correspond to consensus (C), while the filled dot denotes the (stable) self-centered fixed point (S). The small arrows on either side of the nullclines $\dot{\Delta} = 0$ and $\dot{\Sigma} = 0$ (dashed) indicate the local flow of Δ or Σ.

(ii) *Consensus* (C): $\Delta^* = \pm \frac{1}{2}$ and $\Sigma^* = \frac{1}{2}$. One half of all the voters are intrinsically concordant.

To infer the global flow in the Δ-Σ plane, we determine the nullclines $\dot{\Delta} = 0$ and $\dot{\Sigma} = 0$ [given by $\Sigma = \frac{1}{2}$ and $\Sigma = \frac{1}{2}(1+\epsilon) - 2\epsilon\Delta^2$, respectively], and study the linearized rate equations about each fixed point. Both eigenvalues are negative at the self-centered fixed point S, while the eigenvalues have different signs at the consensus fixed points C. Thus if voters have innate preferences, small deviations from the consensus fixed points will grow and the population will be driven to the S fixed point, where each agent tends to align with its innate preference. As the partisanship strength ϵ increases, each individual is more likely to be aligned with its innate preference, but with a concomitant lack of consensus.

For a finite system, however, the only true fixed points of the stochastic dynamics of the PVM are those that correspond to consensus. Since the flow in the Δ-Σ phase plane is driven away from these fixed points, the time to reach consensus should scale exponentially in the population size. We can understand this behavior easily in one dimension because now the dynamics of single domain walls map exactly to the motion of a particle in a random potential (the Sinai model [19]). As illustrated in Fig. 3, strings of consecutive democrats or republicans give rise to potential barriers that domain walls have to surmount to annihilate each other and allow the system to reach consensus. In a system of length N, the mean time for a domain wall to move a distance N therefore scales as e^N [16–19]. Numerical simulations of the PVM

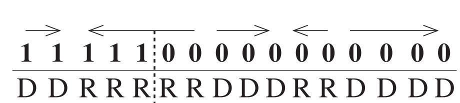

FIG. 3. State of the PVM in one dimension. The letters D and R denote the intrinsic preference of each voter, while the current state of the voters is given the string of **0**s and **1**s. A single domain wall is shown by the dashed line and the bias that it experiences is indicated by the arrows.

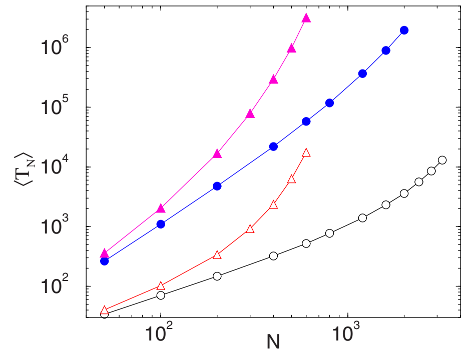

FIG. 4. (Color online) Average consensus time $\langle T_N \rangle$ versus number of voters N for the PVM on (a) the complete graph (open symbols) and (b) a ring (filled symbols). The circles and triangles are simulation data for 10 000 realizations with ϵ=0.05 and 0.15, respectively. The lines are guides to the eyes.

on the complete graph and on the one-dimensional periodic lattice (Fig. 4) are consistent with this prediction. We also checked that qualitatively similar behavior arises when the PVM is generalized to allow for heterogeneity in the flip rate of each voter.

To summarize, we extended the voter model to incorporate the realistic features of heterogeneity and partisanship. Both generalizations are characterized by a much slower approach to consensus than in the classic voter model. When voters are partisan, their individual preferences dominate over collectivism, and it is only by exponentially rare events that consensus can ultimately be achieved. These models offer a step toward the quantitative modeling of social phenomena, such as threshold models of collectivism [10] and social experiments on incentive-driven consensus formation [14]. Particularly interesting behavior seems to arise when the two opinion states are inequivalent; in this situation, partisanship for the unfavorable state may prevent consensus to the favorable state.

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